Exercise 25

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x\to 0} x^2 = 0$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x-0| < \delta \qquad \text{then} \qquad |x^2 - 0| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x|$.

$$|x^2 - 0| < \varepsilon$$

$$|x^2| < \varepsilon$$

$$x^2 < \varepsilon$$

$$\sqrt{x^2} < \sqrt{\varepsilon}$$

$$|x| < \sqrt{\varepsilon}$$

Choose $\delta = \sqrt{\varepsilon}$. Now, assuming that $|x| < \delta$,

$$\begin{aligned} |x^2 - 0| &= |x^2| \\ &= |x||x| \\ &< (\delta)(\delta) \\ &= (\sqrt{\varepsilon})(\sqrt{\varepsilon}) \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 0} x^2 = 0.$$